

Question # 01:

Rectangular co-ordinates of a point are $(3, 3, -3\sqrt{3})$.

- Convert Rectangular co-ordinates to Spherical co-ordinates
- Convert Rectangular co-ordinates to Cylindrical co-ordinates.
- Verify your answer by converting back Rectangular co-ordinates from any one of these, that is, either from Spherical co-ordinates or Cylindrical co-ordinates.

Solution:

Rectangular co-ordinates (x, y, z) to Spherical co-ordinates (ρ, θ, ϕ)


$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{45} \\ \tan \theta &= \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4} \\ \cos \phi &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Rightarrow \phi = \cos^{-1} \left(\frac{-3\sqrt{3}}{\sqrt{45}} \right) = \cos^{-1} \left(\frac{-\sqrt{3}}{\sqrt{5}} \right) \\ &\left(\sqrt{45}, \frac{\pi}{4}, \cos^{-1} \left(\frac{-\sqrt{3}}{\sqrt{5}} \right) \right)\end{aligned}$$

Rectangular co-ordinates (x, y, z) to Cylindrical co-ordinates (r, θ, z)

$$r = \sqrt{x^2 + y^2} = \sqrt{18}$$

$$\tan \theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$z = -3\sqrt{3}$$

$$\left(\sqrt{18}, \frac{\pi}{4}, -3\sqrt{3} \right)$$

Spherical co-ordinates (ρ, θ, ϕ) to Rectangular co-ordinates (x, y, z)

$$x = \rho \sin \phi \cos \theta = \sqrt{45} \cdot \frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}} = 3$$

$$y = \rho \sin \phi \sin \theta = \sqrt{45} \cdot \frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}} = 3$$

$$z = \rho \cos \phi = \sqrt{45} \cdot \frac{-\sqrt{3}}{\sqrt{5}} = -3\sqrt{3}$$

$$(3, 3, -3\sqrt{3})$$

Cylindrical co-ordinates (r, θ, z) to Rectangular co-ordinates (x, y, z)

$$x = r \cos \theta = \sqrt{18} \cdot \cos \frac{\pi}{4} = \sqrt{18} \cdot \frac{1}{\sqrt{2}} = \sqrt{9} = 3$$

$$y = r \sin \theta = \sqrt{18} \cdot \frac{1}{\sqrt{2}} = 3$$

$$z = z = -3\sqrt{3}$$

Rectangular Co-ordinates	Spherical Co-ordinates	Cylindrical Co-ordinates
$(3, 3, -3\sqrt{3})$	$\left(\sqrt{45}, \cos^{-1}\left(\frac{-\sqrt{3}}{\sqrt{5}}\right), \frac{\pi}{4}\right)$	$\left(\sqrt{18}, \frac{\pi}{4}, -3\sqrt{3}\right)$

Question # 02:

Describe the set of all points in xyz-coordinate system at which f is continuous.

$$f(x, y, z) = \sqrt{10-x} \ y \ln z$$

Solution:

$$f(x, y, z) = \sqrt{10-x} \ y \ln z$$

$$D = \{(x, y, z) : x \leq 10, z > 0\}$$

Question # 03:

By considering different path approach, find whether

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{3x^2 + 2y^2} \text{ exist or not.}$$

(Note: In order to get full marks, do all necessary steps)

Solution:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{3x^2 + 2y^2}$$

put $y = mx$

$$= \lim_{(x,mx) \rightarrow (0,0)} \frac{x^2}{3x^2 + 2m^2x^2}$$

$$= \lim_{(x,mx) \rightarrow (0,0)} \frac{x^2}{x^2(3 + 2m^2)}$$

$$= \frac{1}{3 + 2m^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{3x^2 + 2y^2}$$

put $y = x$

$$= \lim_{x \rightarrow 0} \frac{x^2}{5x^2} = \frac{1}{5}$$

Since along different paths we have different limits, hence the limit does not exist.

Question # 01

10

Marks =

Use chain rule to find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \phi}$ if $w = x^2 + y^2 + z^2$ where

$$x = r \cos \theta \sin \phi$$

$$y = r \cos \theta \cos \phi$$

$$z = r \sin \theta$$

Solution:

To find $\frac{\partial w}{\partial r}$, using chain rule, we have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

Substituting the values, we get

$$\frac{\partial w}{\partial r} = (2x)(\cos \theta \sin \phi) + (2y)(\cos \theta \cos \phi) + (2z)(\sin \theta)$$

Putting the values of x, y and z, it yields

$$\begin{aligned} \frac{\partial w}{\partial r} &= 2r \cos^2 \theta \sin^2 \phi + 2r \cos^2 \theta \cos^2 \phi + 2r \sin^2 \theta \\ &= 2r [\cos^2 \theta (\sin^2 \phi + \cos^2 \phi) + \sin^2 \theta] \\ &= 2r (\cos^2 \theta + \sin^2 \theta) \\ &= 2r \end{aligned}$$

Similarly, for $\frac{\partial w}{\partial \phi}$, using chain rule, we get

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \phi}$$

Substituting the values, it becomes

$$\frac{\partial w}{\partial \phi} = (2x)(r \cos \theta \cos \phi) - (2y)(r \cos \theta \sin \phi) + (2z)(0)$$

Putting the values of x and y, it yields

$$\begin{aligned}\frac{\partial w}{\partial \phi} &= 2r^2 \cos^2 \theta \sin \phi \cos \phi - 2r^2 \cos^2 \theta \cos \phi \sin \phi \\ &= 0\end{aligned}$$

Question # 02

10

Marks =

Find the equation of tangent plane to the surface $z = \ln \sqrt{x^2 + y^2}$ at the point $P(-1, 0, 0)$.

Solution:

The given surface is $z = \ln \sqrt{x^2 + y^2}$ which can be written as

$$f(x, y, z) = \ln \sqrt{x^2 + y^2} - z = 0 \text{ ----- (1)}$$

Taking partial derivatives of eq.(1) with respect to x, y and z, we get

$$f_x = \frac{x}{x^2 + y^2}$$

$$f_y = \frac{y}{x^2 + y^2}$$

$$f_z = -1$$

At $P(-1,0,0)$, It becomes

$$f_x(-1,0,0) = -1$$

$$f_y(-1,0,0) = 0$$

$$f_z(-1,0,0) = -1$$

The general form of equation of tangent plane is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

which becomes

$$-(x+1) + 0(y-0) - 1(z-0) = 0$$

$$x + z + 1 = 0$$

which is the required equation of tangent plane.

Question # 03

Marks = 05

Find the critical points of the function given as

$$f(x, y) = x^3 - 3xy - y^3$$

Solution:

The given function is $f(x, y) = x^3 - 3xy - y^3$

Taking partial derivatives of the given function with respect to x and y, we get

$$f_x = 3x^2 - 3y$$

$$f_y = -3x - 3y^2$$

For the critical points, put $f_x = f_y = 0$ which yields

$$x^2 - y = 0 \Rightarrow x^2 = y \text{ ----- (1)}$$

$$-x - y^2 = 0 \Rightarrow x = -y^2 \text{ ----- (2)}$$

Substituting eq. (2) in eq. (1), we get

$$y^4 - y = 0 \Rightarrow y(y^3 - 1) = 0 \Rightarrow y = 0, y = 1$$

Substituting these values of y in eq. (2), we get

$$x = 0, x = -1.$$

Thus, the required critical points are $(0,0)$ and $(-1,1)$

(Note: In order to get full marks, do all necessary steps)

Question No: 1

If $f(x, y) = x^2 + 2y^2 - x^2y$, find the local extrema and saddle points of f .

Solution:

$$f(x, y) = x^2 + 2y^2 - x^2y$$

$$f_x = 2x - 2xy$$

$$f_y = 4y - x^2$$

For critical points

$$f_x = 0 = 2x - 2xy \Rightarrow 2x(1 - y) = 0 \Rightarrow x = 0 \text{ or } y = 1$$

$$f_y = 0 = 4y - x^2 \Rightarrow x^2 = 4y \text{-----(1)}$$

Using $x=0$ in (1) we have $y=0$

$(0,0)$ is critical point.

Using $y=1$ in (1) we have

$$x^2 = 4 \Rightarrow x = \pm 2$$

$(2,1)$ and $(-2,1)$ are critical points.

Now we check on these three critical points for extremum.

$$f_{xx} = 2 - 2y$$

$$f_{yy} = 4$$

$$f_{xy} = -2x$$

$$D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (2 - 2y)(4) - [-2x]^2$$

$$\text{For } (0,0) \ D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = 2(4) - 0 = 8 > 0$$

$$f_{xx} = 2 > 0$$

So f is minimum at $(0,0)$.

$$\text{For } (2,1) \ D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (0)(4) - [-4]^2 = -16 < 0$$

So $(2,1)$ is a saddle point.

$$\text{For } (-2,1) \ D = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (0)(4) - [4]^2 = -16 < 0$$

So $(-2,1)$ is a saddle point.

Question No: 2

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x+y) dy dx$; $a>0$

Solution:

$$\begin{aligned} & \int_0^a \int_0^{\sqrt{a^2-x^2}} (x+y) dy dx \\ &= \int_0^a \left(xy + \frac{y^2}{2} \right) \Big|_0^{\sqrt{a^2-x^2}} dx \\ &= \int_0^a \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right) dx \\ &= \frac{1}{-2} \int_0^a -2x\sqrt{a^2-x^2} dx + \frac{1}{2} \int_0^a (a^2-x^2) dx \\ &= \frac{1}{-2} \frac{(a^2-x^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Big|_0^a + \frac{1}{2} \left(a^2x - \frac{x^3}{3} \right) \Big|_0^a \\ &= \frac{1}{-2} \frac{(a^2-a^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{-2} \frac{(a^2-0)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2} \left(a^3 - \frac{a^3}{3} \right) \\ &= 0 + \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3} \end{aligned}$$

Question No: 3

Evaluate $\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^2} \sin \frac{y}{x} dy dx$

Solution:

$$\begin{aligned}
 & \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^2} \sin \frac{y}{x} dy dx \\
 &= \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left[-\frac{\cos \frac{y}{x}}{\frac{1}{x}} \right]_0^{x^2} dx \\
 &= \int_{\sqrt{\pi}}^{\sqrt{2\pi}} (x - x \cos x) dx \\
 &= \int_{\sqrt{\pi}}^{\sqrt{2\pi}} x dx - \int_{\sqrt{\pi}}^{\sqrt{2\pi}} (x \cos x) dx \\
 &= \frac{x^2}{2} \Big|_{\sqrt{\pi}}^{\sqrt{2\pi}} - [x \sin x]_{\sqrt{\pi}}^{\sqrt{2\pi}} - \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \sin x dx \\
 &= \frac{2\pi}{2} - \frac{\pi}{2} - [x \sin x]_{\sqrt{\pi}}^{\sqrt{2\pi}} - (-\cos x) \Big|_{\sqrt{\pi}}^{\sqrt{2\pi}} \\
 &= \frac{2\pi}{2} - \frac{\pi}{2} - [\sqrt{2\pi} \sin \sqrt{2\pi} - \sqrt{\pi} \sin \sqrt{\pi} + \cos \sqrt{2\pi} - \cos \sqrt{\pi}] \\
 &= \frac{\pi}{2} - \sqrt{2\pi} \sin \sqrt{2\pi} + \sqrt{\pi} \sin \sqrt{\pi} - \cos \sqrt{2\pi} + \cos \sqrt{\pi}
 \end{aligned}$$

(Note: In order to get full marks, do all necessary steps)

Question No: 1

=10

Marks

Evaluate the following integral by converting it to polar coordinates.

$$\iint_R \frac{1}{1+x^2+y^2} dA$$

where R is the sector in the first quadrant bounded by $y = 0$ and $y = x$ and $x^2 + y^2 = 4$.

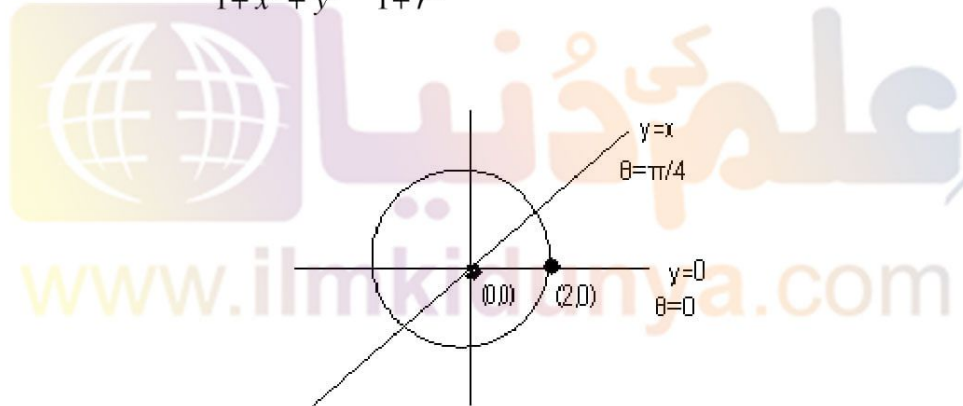
Solution:

Given integral is

$$\iint_R \frac{1}{1+x^2+y^2} dA$$

For conversion to polar coordinates, we have

$$x^2 + y^2 = r^2 \Rightarrow \frac{1}{1+x^2+y^2} = \frac{1}{1+r^2}$$



r ranges from 0 to 2

θ ranges from 0 to $\frac{\pi}{4}$

$$dA = r dr d\theta$$

Thus, the integral in polar coordinates becomes

$$\int_0^{\pi/4} \int_0^2 \frac{1}{1+r^2} r dr d\theta$$

Multiplying and dividing by 2

$$\frac{1}{2} \int_0^{\pi/4} \int_0^2 \frac{2r}{1+r^2} dr d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} [\ln(1+r^2)]_0^2 d\theta \quad \left[\text{By the rule of integration } \int \frac{f'(x)}{f(x)} dx = \ln f(x) \right]$$

$$= \frac{1}{2} \int_0^{\pi/4} (\ln 5 - \ln 1) d\theta$$

$$= \frac{\ln 5}{2} \int_0^{\pi/4} d\theta$$

$$= \frac{\ln 5}{2} [\theta]_0^{\pi/4}$$

$$= \frac{\ln 5}{2} \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi \ln 5}{8}$$

which is the required answer.

Question No: 2 **Marks**
=10

Find the arc length of the curve

$$\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j} + 2\hat{k} \quad 0 \leq t \leq \pi/2$$

Solution:

The given position vector is

$$\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j} + 2\hat{k} \quad \text{where } 0 \leq t \leq \pi/2$$

In parametric form,

$$x = \cos^3 t, \quad y = \sin^3 t, \quad z = 2$$

$$\Rightarrow \frac{dx}{dt} = -3\cos^2 t \sin t, \quad \frac{dy}{dt} = 3\sin^2 t \cos t, \quad \frac{dz}{dt} = 0$$

The arc length of the curve is

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 0} dt \\ &= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} dt \\ &= 3 \int_0^{\pi/2} \cos t \sin t dt \\ &= \left[\frac{3\sin^2 t}{2} \right]_0^{\pi/2} \\ &= \frac{3}{2} \left[\sin^2 \frac{\pi}{2} - \sin^2 0 \right] \\ &= \frac{3}{2} \end{aligned}$$

which is the required answer.

(Note: In order to get full marks, do all necessary steps)

Question # 1:

Evaluate the line integral

$$\int_C (y dx - x^2 dy), \text{ where } C \text{ is the curve } x=t, y=\frac{1}{2}t^2 \text{ where } 0 \leq t \leq 2.$$

Solution:

$$\begin{aligned}
& \int_c (ydx - x^2 dy) \\
&= \int_0^2 ydx - x^2 dy \\
&= \int_0^2 \left[\frac{1}{2} t^2 - t^2 \cdot t \right] dt \\
&= \frac{1}{2} \int_0^2 t^2 dt - \int_0^2 t^3 dt \\
&= \frac{1}{2} \cdot \frac{t^3}{3} \Big|_0^2 - \frac{t^4}{4} \Big|_0^2 \\
&= \frac{4}{3} - 4 \\
&= -\frac{8}{3}
\end{aligned}$$

Question # 2:

Use Green's theorem to evaluate the integral $\oint_c ((x^2 - y)dx + xdy)$, where C is the circle

$x^2 + y^2 = 4$ and c is oriented counterclockwise.

Solution:

$$\oint_c ((x^2 - y)dx + xdy)$$

Using Green's Theorem.

$$\oint_c Pdx + Qdy = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dxdy$$

$$\frac{\partial P}{\partial y} = -1, \frac{\partial Q}{\partial x} = 1$$

$$-\iint_R (-1 - 1) dA$$

$$= 2 \iint_R dA$$

$$= 2 \int_0^{2\pi} \int_0^2 r dr d\theta$$

$$= 4 \int_0^{2\pi} d\theta$$

$$= 8\pi$$

Question # 3:

If $A = (x^4 - y^2 z^2)i + (x^2 + y^2)j - x^3 y^3 z^3 k$, determine curl of A at the point (1,4,-3).

Solution:

$$\text{Curl} A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^4 - y^2 z^2 & x^2 + y^2 & -x^3 y^3 z^3 \end{vmatrix}$$

$$\text{Curl} A = -3x^3 y^2 z^3 i + (3x^2 y^3 z^3 - 2y^2 z) j + (2x + 2yz^2) k$$

$$\text{Curl} A = 1296i - 5088j + 74k \quad \text{at } (1, 4, -3)$$

Question No: 1

=15

Marks

Determine a half range cosine series to represent the function given by

$$f(t) = 2t + 3 \quad 0 < t < 2$$

$$f(t) = f(t + 4)$$

Solution:

To obtain a cosine series, i.e., a series with no sine term involved, we need an even function. Therefore, we assume the wave form to be symmetric about y-axis.

Now, to find the expressions for the Fourier coefficient, we have

$$a_0 = \frac{4}{T} \int_0^{T/2} f(t) dt \quad \text{where } T=4 \text{ is period of the given function.}$$

$$= \frac{4}{4} \int_0^{4/2} (2t + 3) dt$$

$$= \left[\frac{2t^2}{2} + 3t \right]_0^2$$

$$= \{(2)^2 + 3(2)\} - \{(0)^2 + 3(0)\}$$

$$= 4 + 6 + 0$$

$$= 10$$

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega t dt$$

$$= \frac{4}{4} \int_0^{4/2} (2t + 3) \cos n\omega t dt$$

Integrating by parts, we get

$$\begin{aligned}
 a_n &= \left| (2t+3) \frac{\sin n\omega t}{n\omega} \right|_0^2 - \int_0^2 2 \frac{\sin n\omega t}{n\omega} dt \\
 &= \left| (2t+3) \frac{\sin n\omega t}{n\omega} \right|_0^2 - \frac{2}{n\omega} \left| \frac{-\cos n\omega t}{n\omega} \right|_0^2 \\
 &= \frac{7 \sin n\omega 2}{n\omega} + \frac{2}{n^2 \omega^2} [\cos n\omega 2 - 1]
 \end{aligned}$$

Since, $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$

Therefore,

$$a_n = \frac{7 \sin n\pi}{n\omega} + \frac{2}{n^2 \omega^2} [\cos n\pi - 1]$$

$$\sin n\pi = 0,$$

$$\cos n\pi = \begin{cases} 1 & (\text{n is even}) \\ -1 & (\text{n is odd}) \end{cases}$$

So,

$$a_n = \begin{cases} 0 & (\text{n is even}) \\ \frac{-4}{n^2 \omega^2} & (\text{n is odd}) \end{cases}$$

Thus,

$$\begin{aligned}
 f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t \\
 &= 5 - \frac{4}{\omega^2} \left\{ \cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \dots \right\}
 \end{aligned}$$

which is the required fourier series.

Question No: 2

=15

Marks

Verify Stokes' theorem for a hemisphere S defined as $x^2 + y^2 + z^2 = 9$ ($z \geq 0$) where a vector field $F = z^2\hat{i} + 2x\hat{j} - y\hat{k}$ exists over the surface and around its boundary c .

Solution:

Stokes' theorem states that

$$\int_S \text{Curl} F \cdot ds = \oint_c F \cdot dr$$

(a)

$$\begin{aligned} \oint_c F \cdot dr &= \oint_c (z^2\hat{i} + 2x\hat{j} - y\hat{k}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz) \\ &= \oint_c z^2 dx + 2x dy - y dz \end{aligned}$$

For converting it to polar coordinates,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = 0$$

where $r=3$ is the radius of the sphere.

$$x = 3 \cos \theta, \quad y = 3 \sin \theta, \quad z = 0$$

$$dx = -3 \sin \theta d\theta, \quad dy = 3 \cos \theta d\theta, \quad dz = 0$$

$$0 \leq \theta \leq 2\pi$$

Thus, the above integral becomes,

$$\begin{aligned}
& \int_0^{2\pi} 2(3 \cos \theta)(3 \cos \theta d\theta) \\
&= 18 \int_0^{2\pi} \cos^2 \theta d\theta \\
&= 18 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta \\
&= 9 \left[\int_0^{2\pi} d\theta + \int_0^{2\pi} \cos 2\theta d\theta \right] \\
&= 9 [\theta]_0^{2\pi} + 9 \left[\frac{\sin 2\theta}{2} \right]_0^{2\pi} \\
&= 9(2\pi - 0) + \frac{9}{2} (\sin 4\pi - \sin 0) \\
&\oint_c F \cdot dr = 18\pi \quad \text{Eq. (A)}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{curl } F &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2x & -y \end{vmatrix} \\
&= -\hat{i} + 2z\hat{j} + 2\hat{k} \\
\int_s \text{curl } F \cdot d\vec{s} &= \int_s \text{curl } F \cdot \hat{n} ds
\end{aligned}$$

where

$$\begin{aligned}
\hat{n} &= \frac{\nabla s}{|\nabla s|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\
&= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}
\end{aligned}$$

So,

$$\begin{aligned}
\int_s \text{curl } F \cdot d\vec{s} &= \int_s (-\hat{i} + 2z\hat{j} + 2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{3} \right) ds \\
&= \frac{1}{3} \int_s (-x + 2yz + 2z) ds
\end{aligned}$$

For converting it to spherical coordinates, we have

$$x = 3 \sin \phi \cos \theta$$

$$y = 3 \sin \phi \sin \theta$$

$$z = 3 \cos \phi$$

$$ds = 9 \sin \phi d\phi d\theta$$

So, the above integral becomes,

$$\begin{aligned} & \int_s \text{curl} F \cdot d\vec{s} \\ &= \frac{1}{3} \int_s \{-3 \sin \cos \theta + 2(3 \sin \phi \sin \theta)(3 \cos \phi) + 6 \cos \phi\} 9 \sin \phi d\phi d\theta \\ &= 3 \int_s \{-3 \sin^2 \phi \cos \theta + 18 \sin^2 \phi \sin \theta \cos \phi + 6 \cos \phi \sin \phi\} d\phi d\theta \\ &= 3 \int_0^{2\pi} \int_0^{\pi/2} -3 \sin^2 \phi \cos \theta d\phi d\theta + 3 \int_0^{2\pi} \int_0^{\pi/2} 18 \sin^2 \phi \sin \theta \cos \phi d\phi d\theta + 3 \int_0^{2\pi} \int_0^{\pi/2} 6 \cos \phi \sin \phi d\phi d\theta \\ &= I_1 + I_2 + I_3 \quad \text{Eq.(1)} \end{aligned}$$

$$\begin{aligned} I_1 &= -9 \int_0^{2\pi} \int_0^{\pi/2} \sin^2 \phi \cos \theta d\phi d\theta \\ &= -9 \int_0^{2\pi} \int_0^{\pi/2} \frac{1 - \cos 2\phi}{2} \cos \theta d\phi d\theta \\ &= -9 \int_0^{2\pi} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^{\pi/2} \cos \theta d\theta \\ &= -9 \int_0^{2\pi} \left[\frac{\pi}{4} - \frac{\sin \pi}{4} \right] \cos \theta d\theta \\ &= \frac{-9\pi}{4} \int_0^{2\pi} \cos \theta d\theta \\ &= \frac{-9\pi}{4} [\sin \theta]_0^{2\pi} \\ &= \frac{-9\pi}{4} (0) \\ I_1 &= 0 \quad \text{Eq.(2)} \end{aligned}$$

$$\begin{aligned}
I_2 &= 54 \int_0^{2\pi} \int_0^{\pi/2} \sin^2 \phi \sin \theta \cos \phi d\phi d\theta \\
&= 54 \int_0^{2\pi} \left[\frac{\sin^3 \phi}{3} \right]_0^{\pi/2} \sin \theta d\theta \\
&= \frac{54}{3} \int_0^{2\pi} \left[\sin^3 \frac{\pi}{2} - \sin 0 \right] \sin \theta d\theta \\
&= 18 \int_0^{2\pi} \sin \theta d\theta \\
&= -18 [\cos \theta]_0^{2\pi} \\
&= -18 [\cos 2\pi - \cos 0] \\
&= -18(1-1) = 0 \quad \text{Eq.(3)}
\end{aligned}$$

$$\begin{aligned}
I_3 &= 18 \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi d\phi d\theta \\
&= 18 \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} d\theta \\
&= 9 \int_0^{2\pi} (1-0) d\theta \\
&= 9 [\theta]_0^{2\pi} \\
&= 18\pi \quad \text{Eq.(4)}
\end{aligned}$$

Substituting the values of eqs.(2), (3), (4) in eq. (1), we get

$$\int_s \text{curl} F \cdot d\vec{s} = 18\pi \quad \text{Eq.(B)}$$

Hence, from the equality of Eq. (A) and (B), Stokes' theorem is verified.